A new Method to investigate the cases, in which it is possible to solve the difference-differential equation $\partial\partial y(1-axx) - bx\partial x\partial y - cy\partial x^2 = 0$*

Leonhard Euler

§1 Here one could certainly recall the method already explained by me and others at various places, in which the value of y is expressed as an infinite series. For, then in all cases in which this series terminates at some point, one will have a particular integral of the propounded equation; hence it will certainly not be difficult to find the complete integral. But even if one finds infinitely many integrable cases this way, still not all of them are known, but furthermore there are infinitely many other cases that admit a resolution. Therefore, I will propose a completely unique method here, by means of which one will be able to find completely all integrable cases. But this method is of such a nature that having found an arbitrary case admitting a solution from it innumerable others can be deduced.

§2 But two very simple cases, in which the solution succeeds, offer themselves immediately; the one of these case is the case $c = 0$, the other the case

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$b = a$, which two principal cases therefore have to be expanded before all the others.

THE FIRST PRINCIPAL CASE, IN WHICH $c = 0$

§3 Therefore, in this case our equation will be

$$\partial\partial y(1 - axx) = bx\partial x\partial y,$$

which, having put $\partial y = p\partial x$, goes over into this one:

$$\partial p(1 - axx) = bpx\partial x,$$

or

$$\frac{\partial p}{p} = \frac{bx\partial x}{1 - axx},$$

whose integral is

$$\log p = -\frac{b}{2a}\log(1 - axx) + \log C,$$

and so it will be

$$p = C(1 - axx)^{-\frac{b}{2a}} = \frac{\partial y}{\partial x},$$

whence one obtains

$$y = C\int \partial x(1 - axx)^{-\frac{b}{2a}};$$

here it will be helpful to have noted that this value becomes algebraic, as often as $-\frac{b}{2a}$ was an integer number, or $b = -2ia$, while i denotes an arbitrary integer number. But then the value also becomes algebraic, whenever it was $-\frac{b}{2a}$ or $-\frac{3}{2}$ or $-\frac{5}{2}$ or $-\frac{7}{2}$ etc. and hence in general $\frac{b}{a} = 2i + 1$, where i cannot be $= 0$.

The other principal Case, in which $b = a$

§4 Therefore, in this case that our equation, if multiplied by $2\partial y$, will be

$$2\partial y\partial\partial y(1-axx) - 2ax\partial x\partial y^2 - 2cy\partial y\partial x^2 = 0.$$

which is integrable immediately; for, its integral will be

$$\partial y^2(1-axx) - cyy\partial x^2 = C\partial x^2.$$

Therefore, from this equation it will be

$$\partial y\sqrt{1-axx} = \partial x\sqrt{C+cyy},$$

thus, after the substitution it will be

$$\frac{\partial x}{\sqrt{1-axx}} = \frac{\partial y}{\sqrt{C+cyy}}.$$

Thus, this form again contains algebraic cases; to find those let us set $a = -\alpha\alpha$, $c = \gamma\gamma$ and $C = \beta\beta$ such that we have

$$\frac{\partial x}{\sqrt{1+\alpha\alpha xx}} = \frac{\partial y}{\sqrt{\beta\beta+\gamma\gamma yy}},$$

whose integral is

$$\frac{1}{\alpha}\log[\alpha x + \sqrt{1+\alpha\alpha xx}] = \frac{1}{\gamma}\log[\gamma y + \sqrt{\beta\beta+\gamma\gamma yy}] - \frac{1}{\gamma}\log\Delta,$$

whence, going back to numbers, it will be

$$\gamma y + \sqrt{\beta\beta+\gamma\gamma yy} = \Delta[\alpha x + \sqrt{1+\alpha\alpha xx}]^{\frac{\gamma}{\alpha}}.$$

Therefore, having put V for this last expression, it will be

$$V - \gamma y = \sqrt{\beta\beta+\gamma\gamma yy},$$

and taking squares

$$y = \frac{VV-\beta\beta}{2\gamma V}.$$

Therefore, since

$$V = \Delta[\alpha x + \sqrt{1+\alpha\alpha xx}]^{\frac{\gamma}{\alpha}},$$

it will be

$$2\gamma y = \Delta[\alpha x + \sqrt{1+\alpha\alpha xx}]^{\frac{\gamma}{\alpha}} - \frac{\beta\beta}{\Delta}[\alpha x + \sqrt{1+\alpha\alpha xx}]^{-\frac{\gamma}{\alpha}},$$

where $\beta\beta = C$, but the exponents $\frac{\gamma}{\alpha} = \sqrt{-\frac{c}{a}}$, and so, as often as $\sqrt{-\frac{c}{a}}$ was a rational number, the integral will always be algebraic.

§5 Having covered these two principle cases, I will offer two ways to transform the propounded equation into infinitely many others of the same kind such that always an equation of this form

$$\partial\partial Y(1-axx) - Bx\partial x\partial Y - CY\partial x^2 = 0$$

results, since which admits a resolution in the cases $C = 0$ or $B = a$, in the same cases the propounded equation will also be resolvable. Therefore, I will now explain these two transformations.

TRANSFORMATION OF THE FIRST ORDER

§6 I set $y = \frac{\partial v}{\partial x}$, whence because of

$$\partial y = \frac{\partial\partial v}{\partial x} \quad \text{and} \quad \partial\partial y = \frac{\partial^3 v}{\partial x}$$

our equation will take this form:

$$\partial^3 v(1-axx) - bx\partial x\partial\partial v - c\partial x^2\partial v = 0,$$

whose single terms admit an integration; for, it will be

$$\int \partial x^2 \partial v = v\partial x^2,$$

$$\int x\partial x\partial\partial v = x\partial x\partial v - v\partial x^2,$$

$$\int \partial^3 v(1-axx) = \partial\partial v(1-axx) + 2ax\partial x\partial v - 2av\partial x^2.$$

Collecting these parts, our equation will be

$$\partial\partial v(1-axx)-(b-2a)x\partial x\partial v-(c-b+2a)v\partial x^2=0,$$

since which is completely similar to the propounded one, it will integrable in these two cases $c-b+2a=0$ and $b=3a$, or as often as it was $c=b-2a$ or $b=3a$, and having done the integration for each of both cases such that v is expressed in terms of x, then for the propounded equation it will be $y=\frac{\partial v}{\partial x}$; hence it is clear, if the integrals found for v were algebraic, that the value of y will also be algebraic.

§7 If we therefore in like manner set $v=\frac{\partial v'}{\partial x}$, since by means of the preceding operation the letters b and c will go over into $b-2a$ and $c-b+2a$, now this equation will emerge

$$\partial\partial v'(1-axx)-(b-4a)x\partial x\partial v'-(c-2b+6a)v'\partial^2 x=0,$$

which will therefore be integrable, if it was either $b=5a$ or $c=2b-6a$. And having found the values for v' it will be $y=\frac{\partial\partial v'}{\partial x^2}$, the second differentials of v' will give y, of course, and so, if an algebraic value resulted for v', y will also obtain an algebraic value.

§8 If we repeat the same substitution again by putting $v'=\frac{\partial v''}{\partial x}$, for the initial letters b and c we will now have $b-6a$ and $c-3b+12a$, and the resulting equation will be

$$\partial\partial v''(1-axx)-(b-6a)x\partial x\partial v''-(c-3b+12a)v''\partial x^2=0,$$

in which cases the propounded equation therefore also necessarily admits a resolution, since $y=\frac{\partial^3 v''}{\partial x^3}$.

§9 Therefore, if we repeat these same operations continuously, we will always get to equations of the same form; here it will suffice to have noted that both values that we obtained for the letters b and c in each operation, which we want to list up together with the values of y in the following table

		b	c	y
Operation	I.	$b-2a$	$c-b+2a$	$\frac{\partial v}{\partial x}$
	II.	$b-4a$	$c-2b+6a$	$\frac{\partial\partial v'}{\partial x^2}$
	III.	$b-6a$	$c-3b+12a$	$\frac{\partial^3 v''}{\partial x^3}$
	IV.	$b-8a$	$c-4b+20a$	$\frac{\partial^4 v'''}{\partial x^4}$
	$--$	$---$	$--------$	$---$
	$--$	$---$	$--------$	$---$
	$--$	$---$	$--------$	$---$
	i	$b-2ia$	$c-ib+i(i+1)a$	$\frac{\partial^i v^{(i-1)}}{\partial x^i}$

§10 Therefore, it is clear in general that the propounded equation admits a solution, as often as it was

$$\text{either} \quad b = 2ia + a \quad \text{or} \quad c = ib - i(i+1)a,$$

where for i all positive integer numbers can be taken such that hence we obtained two orders of innumerable integrable cases, the second of which are only found by the method of series mentioned initially, the first on the other hand are completely inaccessible by this method.

Transformations of the second Order

§11 As we proceeded through differentials here, let us use integrals now, and first let us certainly put $y = \int z\partial x$, and the propounded equations will become

$$\partial z(1-axx) - bxz\partial x - c\partial x \int z\partial x = 0,$$

which, if differentiated, is reduced to the propounded form

$$\partial\partial z(1-axx)-(b+2a)x\partial x\partial z-(c+b)z\partial x^2=0,$$

which therefore, according to the principle cases, will admit an integration in the cases $c+b=0$ and $b+2a=0$ or $c=-b$ and $b=-a$. Therefore, having found the integrals, it will be $y=\int z\partial x$; hence it is clear, even if these integrals were algebraic, the values of y nevertheless become transcendental.

§12 Further, in like manner, let us set $z=\int z'\partial x$, and since by the preceding operation we obtained $b+2a$ and $c+b$ instead of b and c, we will now get to this equation

$$\partial\partial z'(1-axx)-(b+4a)x\partial x\partial z'-(c+2b+2a)z'\partial x^2=0,$$

which will therefore admit an integration, if it was either $c+2b+2a=0$ or $b+4a=a$ or $c=-2b-2a$ and $b=-3a$. But after having found the value of y from this, we will have $y=\int\partial x\int z'\partial x$, which is reduced to a simple integral sign in such a way that

$$y=x\int z'\partial x-\int z'x\partial x.$$

§13 In like manner, let us further set $z'=\int z''\partial x$, and now we will be led to this equation:

$$\partial\partial z''(1-axx)-(b+6a)x\partial x\partial z''-(c+3b+6a)z''\partial x^2=0,$$

which will therefore be integrable, if it either was $c+3b+6a=0$ or $b+6a=a$, this is, if $c=-3b-6a$ and $b=-5a$; and from these integrals it will be $y=\int\partial x\int\partial x\int z''\partial x$, which value can be reduced from the preceding one, if it is, multiplied by ∂x, integrated again and one writes z'' instead of z'; for, one will obtain

$$y=\frac{1}{2}xx\int z''\partial x-x\int xz''\partial x+\frac{1}{2}\int xxz''\partial x.$$

§14 Therefore, if we continue these operations further, the whole task will reduce to this that the formulas, which will result instead of b and c are formed correctly, and at the same time the values of y are assigned, as the following table will indicate:

	b	c	y
Operat. I	$b+2a$	$c+b$	$\int z\partial x$
II.	$b+4a$	$c+2b+2a$	$\int \partial x \int z'\partial x$
III.	$b+6a$	$c+3b+2a$	$\int \partial x \int x \int z''\partial x$
IV.	$b+8a$	$c+4b+12a$	$\int \partial x \int \partial x \int \partial x \int z'''\partial x$
--	--	--	--
i	$b+2ia$	$c+ib+i(i-1)a$	$\int \partial x \int \partial x \cdots \int z^{(i-1)}\partial x$

§15 From the preceding it is sufficiently clear, how these complicated integrals can be reduced to simpler ones, whence we will just add the following table:

$$\int \partial x \int z'\partial x = x\int z'\partial x - \int z' x\partial x$$
$$\int \partial x \int \partial x \int z''\partial x = \frac{1}{2}\left(xx\int z''\partial x - 2x\int z'' x\partial x + \int z'' xx\partial x\right)$$
$$\int \partial x \int \partial x \int \partial x \int z'''\partial x = \frac{1}{6}\left(x^3\int z'''\partial x - 3xx\int z''' x\partial x + 3x\int z''' xx\partial x - \int z''' x^3\partial x\right)$$
$$\int \partial x \int \partial x \int \partial x \int \partial x \int z^{IV}\partial x = \frac{1}{24}\left(x^4\int z^{IV}\partial x - 4x^3\int z^{IV} x\partial x + 6xx\int z^{IV} xx\partial x - 4x\int z^{IV} x^3\partial + \int z^{IV} x^4\partial x\right)$$

etc.

§16 If we now continued these operations according to the indefinite number i and write B, C, Z instead of b, c, z, the resulting equation will be

$$\partial\partial Z(1-axx) - Bx\partial x\partial Z - CZ\partial x^2 = 0,$$

where it will be, as we already indicated,

$$B = b + 2ia \quad \text{and} \quad C = c + ib + i(i-1)a;$$

therefore, this equation will admit an integration, as often as it was

$$\begin{aligned} &\text{either} \quad C = 0 \quad \text{this is} \quad c = -ib - i(i-1)a, \\ &\text{or} \qquad B = a \quad \text{this is} \quad b = -(2i-1)a; \end{aligned}$$

these formulas differ from those, that we found above for the first order of transformations, just in that regard that here the letter i will take a negative value; hence one has the following

General Conclusion

§17 If the letter i denotes all integer either positive or negative numbers here, the propounded difference-differential equation

$$\partial\partial y(1 - axx) - bx\partial x\partial y - cy\partial x^2 = 0$$

will always admit an integration or resolution, as often as it was

$$1) \quad c = ib - i(i+1)a$$

or

$$2) \quad b = (2i+1)a,$$

where we can assert with confidence that completely all resolvable cases are contained in these two forms such that absolutely no case that admits an integration can be exhibited, which is not comprehended in the one of these two formulas, whereas the method proceeding through series that we mentioned earlier only shows the first integrable cases such that hence an infinite number of likewise resolvable cases are excluded.

Corollary 1

§18 Transform the propounded equation into a differential of first degree by putting $y = e^{\int u\partial x}$, and we will get to this equation:

$$\partial u + uu\partial x - \frac{bux\partial x + c\partial x}{1 - axx} = 0,$$

which therefore will also admit an integration in the cases in which

$$\text{either} \quad b = (2i+1)a \quad \text{or} \quad c = ib - i(i+1)a,$$

while i denotes an arbitrary either positive or negative integer.

COROLLARY 2

§19 Therefore, if one further sets

$$u = (1-axx)^n v,$$

after, for the sake of brevity, having put $n = -\frac{b}{2a}$, one will get to this equation which is to be referred to the RICCATIAN kind:

$$(1-axx)^n \partial v + (1-axx)^{2n} vv\partial x = \frac{c\partial x}{1-axx},$$

which divided by $(1-axx)^n$ goes over into this one:

$$\partial v + (1-axx)^n vv\partial x = \frac{c\partial x}{(1-axx)^{n+1}},$$

which will therefore admit an integration in the same cases.

COROLLARY 3

§20 Thus, if we take $a = 0$, this equation will arise:

$$\partial u + uu\partial x = bux\partial x + c\partial x,$$

which therefore will be integrable, if it either was $b = 0$ or $c = ib$, the first of which cases is obvious per se, since then it will be

$$\partial x = \frac{\partial u}{c - uu}.$$

But this form can be expressed more conveniently by putting

$$u = \frac{1}{2}bx + v, \quad \text{whence} \quad \partial v + vv\partial x = \left(c - \frac{1}{2}b\right)\partial x + \frac{1}{4}bbxx\partial x,$$

or by putting $b = 2f$ such that

$$\partial v + vv\partial x = (c-f)\partial x + ffxx\partial x,$$

and this equation will be integrable, as often as it was $c = 2if$ such that the following equation always admits an integration:

$$\partial v + vv\partial x = (2i-1)f\partial x + ffxx\partial x,$$

whatever either positive or negative integer number is taken for i; this means, if in the penultimate term f is multiplied by an arbitrary either positive or negative odd number, which cases will be the more obscure the greater the number i is taken; and there even seems to be no other way to find the integrals than to return to the propounded differential equation of second degree and perform the same operation we taught above. Nevertheless, I observed that all those cases can also be derived immediately from the equation by means of continued fractions. For, if this equation was propounded:

$$\partial v + vv\partial x = g\partial x + ffxx\partial x,$$

the value of v can be expressed by a continued fraction in two ways. For, on the one hand

$$v = fx + \cfrac{g-f}{2fx + \cfrac{g-3f}{2fx + \cfrac{g-5f}{2fx + \text{etc.}}}}$$

On the other hand

$$v = -fx - \cfrac{(g-f)}{2fx + \cfrac{g+3f}{2fx + \cfrac{g+5f}{2fx + \cfrac{g+7f}{2fx + \text{etc.},}}}}$$

the first of which terminates, as often as it was $g = (2i+1)f$, the second on the other hand, as often as it was $g = -(2i+1)f$, which are the integrable cases found before.